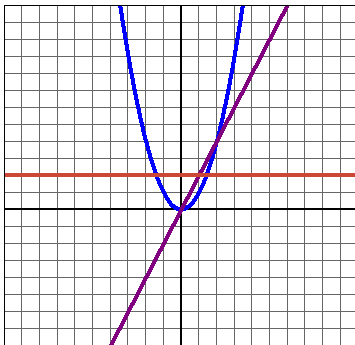


Calculus 140, section 4.7 Concavity and Inflection Points

notes by Tim Pilachowski

Reminder: You will not be able to use a graphing calculator on tests!



Theory Example: Consider the graph of $y = x^2$ pictured to the left along with its derivatives $y' = 2x$ and $y'' = 2$.

Interval	$y = x^2$ is...	$y' = 2x$ is...	$y'' = 2$ is...

The second derivative is the slope of the first derivative, and tells us how the first derivative is changing, i.e. how the slope of the function is itself changing. In the graph of $y = x^2$ above, the slope (first derivative) is negative on the interval $-\infty < x < 0$.

Note that the slope of the parabola is becoming less steep (more shallow) as x approaches 0. Another way to say the same thing is that the slope of the parabola (first derivative), while still negative, is becoming less negative as x approaches 0, until the curve hits $x = 0$, at which point the slope of the parabola (first derivative) equals 0.

On the interval $0 < x < \infty$ the slope of the parabola (first derivative) is positive. Note, too, that the slope of the parabola is becoming steeper (i.e. the first derivative is becoming ever-larger positive numbers) as x approaches ∞ .

The slope of the curve = the first derivative is progressing in this way:

very negative < less negative < zero < small positive < large positive
 slope of curve is always increasing = value of first derivative is always increasing
 = slope of first derivative is always positive = value of second derivative is always positive

Vocabulary: The graph of $y = x^2$ is **concave up** over its entire domain.

There is a connection between the **concavity** of a function (Definition 4.12) and its second derivative.

If we applied a similar process to $y = \sqrt{x}$ we would find that $y'' < 0$ on $(0, \infty)$, and we'd say that the graph of $y = \sqrt{x}$ is **concave down** over its entire domain.

Theorem 4.13: "Assume that f'' exists on an open interval I .

- a. If $f'' > 0$ for all x in I , then the graph of f is concave upward on I .
- b. If $f'' < 0$ for all x in I , then the graph of f is concave downward on I ."

A point on a graph where the concavity of the curve changes (from concave down to concave up, or vice versa) is called a **point of inflection** (Definition 4.14).

By implication (think about what separates positive and negative numbers on a number line), if a point $(c, f(c))$ is a point of inflection, then $f''(c) = 0$.

Important: This is a one-way conditional logic statement! At the point $(0, 0)$ on the graph of $f(x) = x^4$, both $f'(0) = 0$ and $f''(0) = 0$, but since the concavity does not change, $(0, 0)$ is not a point of inflection.

4.5 Example A revisited: Consider $f(x) = x^3 - 3x^2 - 9x + 1$.

first derivative: $f'(x) = 3x^2 - 8$

second derivative:

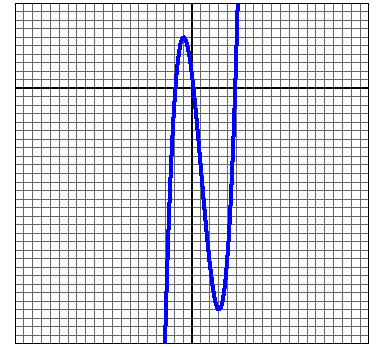
critical numbers:

critical points:

interval(s) concave up:

interval(s) concave down:

point(s) of inflection:



4.5 Example B revisited: Consider the function $f(x) = \frac{x^3}{e^x}$.

first derivative: $f'(x) = \frac{3x^2 - x^3}{e^x}$

second derivative:

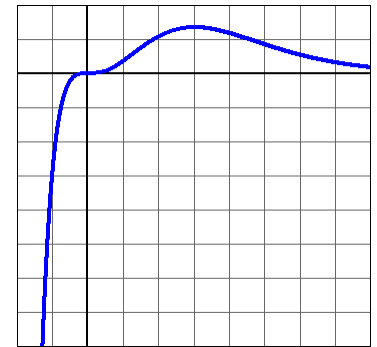
critical numbers:

critical points:

interval(s) concave up:

interval(s) concave down:

point(s) of inflection:



4.5 Example C revisited: Given $f(x) = \cos x + \frac{\sqrt{2}}{2}x$, determine values c where $f''(x)$ changes from negative to positive, or from positive to negative.

first derivative:

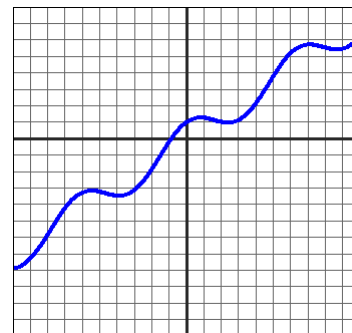
second derivative:

critical numbers:

interval(s) concave up:

interval(s) concave down:

point(s) of inflection:



4.5 Example D revisited: Consider $f(x) = \frac{3x+1}{x-2}$.

first derivative: $f'(x) = \frac{-7}{(x-2)^2}$

second derivative:

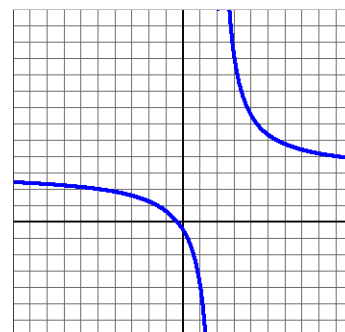
critical numbers:

critical points:

interval(s) concave up:

interval(s) concave down:

point(s) of inflection:



4.5 Example E revisited: Consider $f(x) = 2x + \frac{2}{x} - 1 = 2x + 2x^{-1} - 1$.

first derivative: $f'(x) = 2 - \frac{2}{x^2}$

second derivative:

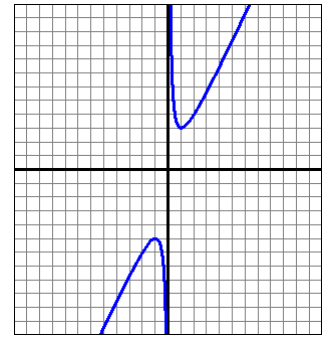
critical numbers:

critical points:

interval(s) concave up:

interval(s) concave down:

point(s) of inflection:



4.5 Example F revisited: Consider $f(x) = \frac{10 \ln x}{x}$.

first derivative: $f'(x) = \frac{10 - 10 \ln x}{x^2}$

second derivative:

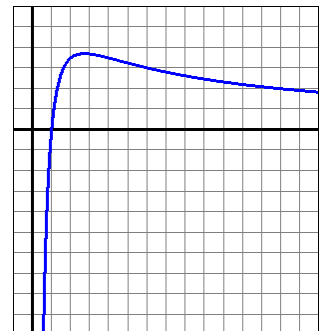
critical numbers:

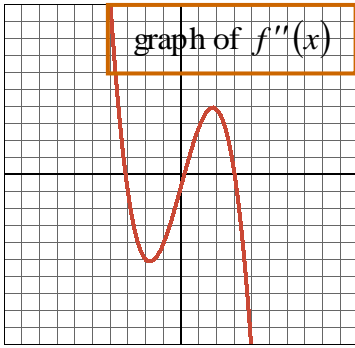
critical points:

interval(s) concave up:

interval(s) concave down:

point(s) of inflection:





4.5 Example G, part 2: Without knowing the function itself, describe the behavior of its graph only using information provided by its second derivative. The graph to the left is a graph of $f''(x)$.

Since $f''(x) = 0$ at $x = -3$, $x \cong 0.1$, and $x = 3$, we'll look in those places for points of inflection. We can determine whether f is concave up or down by determining where f'' is positive or negative.

interval	$x < -3$	$x = -3$	$-3 < x < 0.1$	$x \cong 0.1$	$0.1 < x < 3$	$x = 3$	$3 < x$
value of f''							
f is concave...							

interval(s) increasing:

interval(s) decreasing:

points of inflection:

Using this information, along with information from Lecture 4.5, we can draw a possible graph for f , which may look something like this:

